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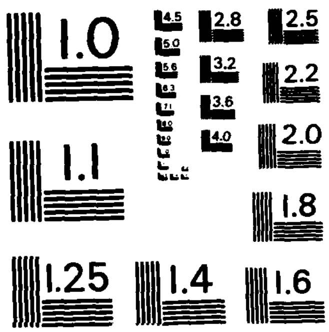
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Bounds on Distributions Arising in Order Restricted Inference: The Partially Ordered Case

ABSTRACT

In testing hypotheses involving order restrictions on a collection of parameters, distributions arise which are mixtures of chi-squared or beta distributions. In general, the mixing coefficients are quite intractable even for a moderate number of populations. Stochastic upper and lower bounds are obtained for mixtures which arise in Bartholomew's tests for homogeneity of normal means with the alternative restricted by a quasi ordering. These bounds are applicable in the dual-testing situation, that is in testing the order restriction as a null hypothesis; in testing order restrictions in exponential families, Poisson intensities and multinomial parameters; and in some nonparametric settings. ~~They~~ These can also be applied to obtain the least favorable configuration for testing equality of two multinomial populations with a stochastic ordering alternative.

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1 INTRODUCTION. Suppose one is interested in testing hypotheses about a vector of normal means, $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, and it is known apriori that these means satisfy certain restrictions, such as $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_k$ or $\mu_i \geq 0$ for $i = 1,2,\ldots,k$. Likelihood ratio tests (LRTs) which make use of such order restrictions have been derived under various assumptions on the covariance structure.

Bartholomew (1959, 1961) considered testing homogeneity subject to the restriction that the μ_i satisfy a partial ordering. Let $<<$ be a quasi order on $\Gamma = \{1,2,\ldots,k\}$, that is $<<$ is reflexive, ie. $i << i$ for all $i \in \Gamma$, and transitive, ie. $i << j << \ell$ implies $i << \ell$. (A partial order is also antisymmetric, ie. $i << j << i$ implies $i = j$.) If $\mu_i \leq \mu_j$ for all $i << j$, then μ is isotonic with respect to (wrt) $<<$. Since $k = 1$ is not interesting, we suppose $k \geq 2$. For $j = 1,2,\ldots,n_i$ and all $i \in \Gamma$, let

(A1) X_{ij} be independent, $X_{ij} \sim N(\mu_i, \sigma_i^2)$ and $\sigma_i^2 = a_i \sigma_0^2$ with a_i known and $\sigma_0^2 = 1$ if known.

Bartholomew studied the LRT of H_0: $\mu_1 = \mu_2 = \ldots = \mu_k$ versus $H_1 - H_0$ with H_1: μ is isotonic. He showed that the null distribution is a mixture of chi-squared or beta distributions depending on whether σ_0^2 is known or not. Under a more general assumption about the covariance structure, Perlman (1969) considered the LRT of H_1 versus H_2: μ is not isotonic and obtained bounds for the significance level of the test. Assuming (A1), Robertson and Wegman (1978) proved that H_0 is least favorable for testing H_1 versus H_2 and showed that the distribution of the LRT statistic, under H_0, is again a

mixture of chi-squared or beta distributions with the same mixing coefficients as in Bartholomew's test.

The mixing coefficients depend upon the quasi order, <<, and $w = (w_1, w_2, \ldots, w_k)$ where $w_i = n_i/a_i$ for $i \in \Gamma$. Even in the case in which << is a total order (ie. $i << j$ or $j << i$ for all $i, j \in \Gamma$), these coefficients are quite intractable if the weights, w_i, are not equal and $k > 5$, cf. Barlow et al. (1972, p. 142). (The equal-weights mixing coefficients for a total order are given in their Table A.5.) For the totally ordered case with unequal weights and σ_0^2 known or unknown, the distributions, under H_0, of the LRT statistics for H_0 versus H_1 and H_1 versus H_2 have been studied. Approximations were developed by Chase (1974), Syskind (1976) and Robertson and Wright (1983). Upper and lower bounds for their significance levels are given in Robertson and Wright (1982).

For the partially ordered case, less is known about the null distributions for these tests. An iterative scheme for computing the mixing coefficients is discussed in Barlow et al. (1972, p. 139). They also give exact expressions for these coefficients for arbitrary weights, $k \leq 4$ and the following partial orders:

(1) the simple tree; $1 << i$ for $i = 2, 3, \ldots, k$ (the equal-weights coefficients for a simple tree are given in their Table A.6) and

(2) the simple loop; $1 << i << k$ for $i = 2, 3, \ldots, k-1$.

There are other partial orders that are of practical importance, too. For instance, if μ_{ij} is the mean response of a dependent variable with one independent variable at level i and the other at level j and μ_{ij} is nondecreasing in i with j held fixed and vice versa, then $\mu = (\mu_{ij})$ is

isotonic wrt <<, the coordinatewise partial ordering. (This example is discussed in Brunk, Ewing and Utz (1957) and Hanson, Pledger and Wright (1973).) The unimodal ordering, $\mu_1 \le \mu_2 \le \ldots \le \mu_h \ge \mu_{h+1} \ge \ldots \ge \mu_k$ with $1 < h < k$ is also of interest. There is much to be learned about the mixing coefficients for such partial orders. Robertson and Wright (1985) and Wright and Tran (1985) give approximations and bounds for the case of a simple tree.

In Section 2, we show how the quasi ordered case is reduced to the partially ordered case and obtain sharp bounds for the significance levels of the LRTs of H_0 versus H_1 and H_1 versus H_2 which are independent of the weights, w_i. In Section 3, some applications of these bounds are discussed.

Kudo (1963) considered a closely related problem, that of testing H_0': $\theta = 0$ versus $H_1'-H_0'$ with H_1': $\theta \ge 0$, where $\theta \ge 0$ means each coordinate of θ is nonnegative. He derived the LRT based on a random sample from a multivariate normal distribution, ie.

(A2) $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{ip})$ are i.i.d. $N(\theta, \Sigma)$ for $i = 1, 2, \ldots, n$ with

Σ nonsingular,

for the case of known Σ. Neusch (1966) treated this problem with $\Sigma = \sigma_0^2 \Lambda$ and Λ known. (See the comments in Section 12 of Shorack (1967).) Perlman (1969) considered the case of unknown, nonsingular Σ. In fact, Perlman derived the LRTs for $\theta = 0$ versus $\theta \in C-\{0\}$ and $\theta \in C$ versus $\theta \notin C$ with C a positively homogeneous, one-sided subset of R^p (see that paper for definitions) and obtained sharp bounds for the significance levels of these tests. Some of the testing problems discussed earlier can be placed in this framework. For instance, for the simple tree with variances known, set $n =$

1, $p = k-1$, $Y_{1j} = \bar{X}_{j+1} - \bar{X}_1$ for $j = 1,2,\ldots,p$ and C the p-dimensional positive orthant. The total order and unimodal order can be treated similarly. However, if the variances are unknown and the sample sizes are not the same, or if this differencing yields a nonsingular Σ (see the discussion of the simple loop in Barlow et al. (1972, p. 176)), then this technique does not seem to work. On the other hand, (A2) includes situations not included in (A1). The bounds given here are, for some partial orders, tighter than the bounds obtained from Perlman's work, but that is because they depend on the special covariance structure implied by (A1). Perlman shows that for any C, which contains a p-dimensional open set, his bounds are sharp.

2 TAIL PROBABILITY BOUNDS. Throughout this section we assume that (A1) holds unless stated otherwise. If σ_0^2 is known the LRT of H_0 versus H_1 rejects for large values of

$$\bar{\chi}_{01}^2 = \sum_{i=1}^{k} w_i(\bar{\mu}_i - \hat{\mu})^2$$

with $\hat{\mu} = \sum_{i=1}^{k} w_i\bar{X}_i / \sum_{i=1}^{k} w_i$ and $\bar{\mu} = E_W(\bar{X}|H_1)$, the projection of $\bar{X} = (\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_k)$ onto H_1 with respect to (wrt) the distance determined by the inner product $(x,y)_w = \sum_{i=1}^{k} w_i x_i y_i$. Furthermore, under H_0

$$(2.1) \quad P_w[\bar{\chi}_{01}^2 \geq c] = \sum_{\ell=1}^{k} P(\ell,k;w)P[\chi_{\ell-1}^2 \geq c]$$

with $\chi_0^2 \equiv 0$ and $P(\ell,k;w)$ the probability, under H_0, that $E_w(\bar{X}|H_1)$ has exactly ℓ distinct values (cf. Barlow et al. (1972, Chapter 3)). Of course,

$\bar{\chi}^2_{01}$ can be written as $||\bar{\mu}-\hat{\mu}||^2_w$, in which $\hat{\mu}$ also denotes a k-dimensional vector with each component equal to $\hat{\mu}$. Barlow et al. also discusses the LRT of H_0 versus H_1 if σ_0^2 is unknown. It rejects H_0 for large values of

$$\bar{E}^2_{01} = \sum_{i=1}^k w_i(\bar{\mu}_i-\hat{\mu})^2/\sum_{i=1}^k a_i^{-1}\sum_{j=1}^{n_i}(X_{ij}-\hat{\mu})^2.$$

Under H_0,

$$(2.2)\quad P_w[\bar{E}^2_{01} \geq c] = \sum_{\ell=1}^k P(\ell,k;w)P[B_{\frac{1}{2}(\ell-1),\frac{1}{2}(N-\ell)} \geq c]$$

with $B_{a,b}$ a beta variable with parameters a and b ($B_{0,b} \equiv 0$) and $N = \sum_{i=1}^k n_i$.

If σ_0^2 is known, the LRT of H_1 versus H_2: μ is not isotonic rejects for large values of

$$\bar{\chi}^2_{12} = \sum_{i=1}^k w_i(\bar{X}_i-\bar{\mu}_i)^2 = ||\bar{X}-\bar{\mu}||^2_w,$$

H_0 is least favorable within H_1, and under H_0,

$$(2.3)\quad P_w[\bar{\chi}^2_{12} \geq c] = \sum_{\ell=1}^k P(\ell,k;w)P[\chi^2_{k-\ell} \geq c].$$

If σ_0^2 is unknown, the LRT of H_1 versus H_2 rejects for large values of

$$\bar{E}^2_{12} = \sum_{i=1}^k w_i(\bar{X}_i-\bar{\mu}_i)^2/[\sum_{i=1}^k w_i(\bar{X}_i-\bar{\mu}_i)^2 + Q]$$

with $Q = \sum_{i=1}^k a_i^{-1}\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_i)^2$. H_0 is again least favorable within H_1 and under H_0,

(2.4) $P_w[\bar{E}_{12}^2 \geq c] = \sum_{\ell=1}^{k} P(\ell,k;w)P[B_{\frac{1}{2}(k-\ell),\frac{1}{2}(N-k)} \geq c]$.

(See Robertson and Wegman (1978).)

We seek upper and lower bounds for the significance levels given by (2.1)-(2.4). The techniques from Perlman (1969) and Robertson and Wright (1982) will be combined. In the latter reference, it was shown that for the total order, $1 << 2 << \ldots << k$, an upper (lower) bound for (2.1) and (2.2) ((2.3) and (2.4)) is obtained by replacing $P(\ell,k;w)$ by $A(\ell,k) = \binom{k-1}{\ell-1}2^{-k+1}$, $1 \leq \ell \leq k$, in those expressions, and a lower (upper) bound for (2.1) and (2.2) ((2.3) and (2.4)) is obtained by replacing $P(\ell,k;w)$ by $B(\ell,k) = \frac{1}{2} I_{\{1,2\}}(\ell)$ with $I_A(\ell) = 1$ if $\ell \in A$ and $I_A(\ell) = 0$ if $\ell \notin A$. They also established the sharpness of these bounds. The bounds resulting from the $B(\ell,k)$ were obtained by Perlman (1969) and the mixing coefficients, $A(\ell,k)$, were encountered by Dykstra and Robertson (1982, 1983) in related testing problems.

If Y is a binomial variable with parameters $k-1$ and $\frac{1}{2}$, then the probability function of $Y+1$ is $A(\ell,k)$, and so we refer to the $\{A(\ell,k)\}$ as shifted binomial probabilities. Because the bounds of interest here involve convolutions of shifted binomial probabilities, we give the following definition. For positive integers $b \leq k$, set $A_b(\ell,k) = \binom{k-b}{\ell-b}2^{-k+b}$ for $\ell = 1,2,\ldots,k$. (Of course, $A_b(\ell,k) = 0$ for $\ell < b$ and $A(\ell,k) \equiv A_1(\ell,k)$.)

The LRT statistics and their null distributions given in this section were derived for partial orders. However, if $\bar{\mu}$ is understood to be the projection of $\bar{X}$ onto the cone corresponding to a quasi order, $<<$, then the

arguments given in the references above show that the LR procedure gives rise to the same test statistics, $\bar{\chi}^2_{01}, \bar{\chi}^2_{12}, \bar{E}^2_{01}$ and $\bar{E}^2_{12}$. Dykstra (1981) discusses the computation of $\bar{\mu}$ for an arbitrary quasi order. The argument given to show that H_0 is least favorable within H_1 in the partially ordered case is also applicable for an arbitrary quasi order. In the next remark, we show that (2.1)-(2.4) are also valid for quasi orders. The proof consists of reducing << to a partial order and applying the known results. This reduction also shows that we only need to obtain bounds for the partially ordered case.

If << is a quasi order, then $i \approx j$ if and only if $i << j << i$ is an equivalence relation and induces a natural partial order, <<*, on the equivalence classes $\{e_\alpha : \alpha = 1,2,\ldots,k^*\}$. We identify α with e_α, think of <<* as a partial order on $\{1,2,\ldots,k^*\}$ and let H_1^* denote the vectors in R^{k^*} isotonic wrt <<*. For $1 \le \alpha \le k^*$, set $w^*_\alpha = \sum_{i\varepsilon e_\alpha} w_i$ and $\bar{X}^*_\alpha = \sum_{i\varepsilon e_\alpha} \bar{X}_i / w^*_\alpha$; let $\bar{\mu}^*$ denote $E_{w^*}(\bar{X}^* | H_1^*)$; and let $P(\ell,k^*;w^*)$ denote the probability under homogeneity, that $\bar{\mu}^*$ contains exactly ℓ distinct values. (Of course $P(\ell,k^*;w^*) = 0$ for $k^* < \ell \le k$.)

<u>Remark</u>. Under homogeneity, (2.1)-(2.4) are valid for quasi orders and their right hand sides are unchanged if $P(\ell,k;w)$ is replaced by $P(\ell,k^*;w^*)$.

<u>Proof</u>. Clearly $\hat{\mu}^* = \sum_{\alpha=1}^{k^*} w^*_\alpha \bar{X}^*_\alpha / \sum_{\alpha=1}^{k^*} w^*_\alpha = \hat{\mu}$. Since $E_w(\bar{X} | H_1)$ is constant on each e_α, it must agree with $E_{w^*}(\bar{X}^* | H_1^*)$ on these equivalence classes. Hence,

$P(\ell,k;w) = P(\ell,k^*;w^*)$ for $\ell \le k$, and consequently, the second conclusion is established. Next, we observe that

$$(2.5)\quad \bar{\chi}^2_{01} \equiv \sum_{i=1}^{k} w_i(\bar{\mu}_i-\hat{\mu})^2 = \sum_{\alpha=1}^{k^*} \sum_{i\varepsilon e_\alpha} w_i(\bar{\mu}^*_\alpha - \hat{\mu}^*)^2 = \sum_{\alpha=1}^{k^*} w^*_\alpha(\bar{\mu}^*_\alpha - \hat{\mu}^*)^2 \equiv U$$

and since $\bar{\mu}$ is constant on e_α,

$$(2.6)\quad \bar{\chi}^2_{12} \equiv \sum_{i=1}^{k} w_i(\bar{X}_i - \bar{\mu}_i)^2 = Q' + V$$

with $Q' \equiv \sum_{\alpha=1}^{k^*} \sum_{i\varepsilon e_\alpha} w_i(\bar{X}_i - \bar{X}^*_\alpha)^2$ and $V \equiv \sum_{\alpha=1}^{k^*} w^*_\alpha(\bar{X}^*_\alpha - \bar{\mu}^*_\alpha)^2$. We assume H_0 for the remainder of the proof. It follows from (2.5) and Bartholomew's work that (2.1) holds for quasi orders. Clearly, Q, Q' and $\bar{X}^*$ are independent and $Q' \sim \chi^2(k-k^*)$. Conditioning on Q' and applying Corollary (4.2) of Robertson and Wegman (1978), we see that

$$P_w[\bar{\chi}^2_{12} \ge c] = \sum_{\ell=1}^{k^*} P(\ell,k^*;w^*)P[\chi^2_{k-\ell} \ge c]$$

and so (2.3) also holds for quasi orders. Rewriting $\bar{E}^2_{01}$ as $\bar{\chi}^2_{01}/(Q + \bar{\chi}^2_{01} + \bar{\chi}^2_{12})$, recalling that $\bar{\chi}^2_{01} = U$ and $\bar{\chi}^2_{12} = V+Q'$ and noting that $Q + Q' \sim \chi^2(N-k^*)$, we see that $\bar{E}^2_{01}$ has the same distribution as the corresponding test statistic for the partial order, <<*, with a total sample size of N. Thus, applying Bartholomew's results on $\bar{E}^2_{01}$, we see that (2.2) holds for an arbitrary quasi order. Since $0 \le \bar{E}^2_{12} \le 1$, we consider $c \in (0,1)$ and note that

$$P[\bar{E}^2_{12} \geq c] = P[\bar{\chi}^2_{12}/Q \geq c/(1-c)] = P[V \geq cQ/(1-c) - Q'].$$

Applying the result of Robertson and Wegman for V to the above probability, conditioned on Q and Q', we obtain

$$P[\bar{E}^2_{12} \geq c] = \sum_{\ell=1}^{k^*} P(\ell,k^*;w^*)P\left[\frac{\chi^2_{k^*-\ell} + Q'}{\chi^2_{k^*-\ell} + Q' + Q} \geq c\right]$$

with $\chi^2_{k^*-\ell}$, Q and Q' independent. Thus, (2.4) holds for quasi orders, too. The proof is completed.

Because the null distribution of $\bar{\chi}^2_{01}$ and $\bar{E}^2_{01}$ for a quasi order, <<, is the same as for the associated partial order, <<*, bounds for a partial order are also bounds for any quasi order that reduces to the given partial order. Considering the proof of the Remark, it is clear that if $\sum_{\ell=1}^{k^*} a_\ell P[\chi^2_{k^*-\ell} \geq c]$ provides a bound, either upper or lower, for the tail probability of $\bar{\chi}^2_{12}$ under H_0 for the partial order <<*, then

$$\sum_{\ell=1}^{k^*} a_\ell P[\chi^2_{k-\ell} \geq c] \quad \left(\sum_{\ell=1}^{k^*} a_\ell P[B_{\frac{1}{2}(k-\ell),\frac{1}{2}(N-k)} \geq c]\right)$$

is an analogous bound for the tail probabilities of $\bar{\chi}^2_{12}(\bar{E}^2_{12})$ under H_0 for the quasi order <<. Hence, we need only obtain bounds for the partially ordered case.

We suppose that << is a partial order and consider upper (lower) bounds for (2.1) and (2.2) ((2.3) and (2.4)). First, we need the following.

Definitions. Two distinct elements $i,j \in \Gamma$ are comparable if $i << j$ or $j << i$. Otherwise, they are noncomparable.

A subset of Γ is a chain (antichain) if each pair of distinct elements in the subset is comparable (noncomparable).

The breadth, b, of Γ is defined to be the maximal cardinality of any antichain in Γ, ie. $b = \max\{\text{card.}(A): A \text{ an antichain in } \Gamma\}$.

The following result is given in Crawley and Dilworth (1973, p. 3).

Theorem 2.1. Let $(\Gamma, <<)$ be a partially ordered set with breadth b. Γ can be written as the disjoint union of b sets which are chains wrt $<<$.

Theorem 2.2. Let $(\Gamma, <<)$ be a partially ordered set with breadth b, and let $\mu \in H_0$. For any w with $w_i > 0$ for each $i \in \Gamma$,

(2.7a) $P_w[\bar{\chi}^2_{01} \geq c] \leq \sum_{\ell=1}^{k} A_b(\ell,k) P[\chi^2_{\ell-1} \geq c]$,

(2.7b) $P_w[\bar{E}^2_{01} \geq c] \leq \sum_{\ell=1}^{k} A_b(\ell,k) P[B_{\frac{1}{2}(\ell-1),\frac{1}{2}(N-\ell)} \geq c]$,

(2.7c) $P_w[\bar{\chi}^2_{12} \geq c] \geq \sum_{\ell=1}^{k} A_b(\ell,k) P[\chi^2_{k-\ell} \geq c]$ and

(2.7d) $P_w[\bar{E}^2_{12} \geq c] \geq \sum_{\ell=1}^{k} A_b(\ell,k) P[B_{\frac{1}{2}(k-\ell),\frac{1}{2}(N-k)} \geq c]$.

Proof. Using (7.6) and (7.7) of Barlow et al. and the fact that $H_1 \supset H_0$, it is easily shown that $||\bar{X} - \hat{\mu}||^2_w = ||E_w(\bar{X}|H_1) - \hat{\mu}||^2_w + ||\bar{X} - E_w(\bar{X}|H_1)||^2_w$. Furthermore, if H_1' corresponds to some quasi order on Γ and $H_1' \supset H_1$, then $||\bar{X} - E_w(\bar{X}|H_1)||^2_w \geq ||\bar{X} - E_w(\bar{X}|H_1')||^2_w$ and consequently, $||E_w(\bar{X}|H_1) - \hat{\mu}||^2_w \leq ||E_w(\bar{X}|H_1') - \hat{\mu}||^2_w$. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_b$ be disjoint chains with $\bigcup_{\alpha=1}^{b} \Gamma_\alpha = \Gamma$; let $<<'$ be defined by $i <<' j$ provided $i, j \in \Gamma_\alpha$ for some α and $i << j$; and let

k; the other is similar. Since k is an interior element there exists $c \neq k$ with $k \ll c$. Of course, $c \in A$ or $c \in B$, but either case contradicts $A \hat{+} B = \Gamma - \{k\}$.

It is easy to show that $\Gamma - \{k\}$ has $i-1$ interior and e exterior elements. By the inductive hypothesis $P(\ell,k-1;w^{(k-1)})$ can be made close to $A(\ell,e)$. The desired conclusion follows readily, and the proof is completed.

3 APPLICATIONS. The bounds given in Section 2 provide conservative critical values or p-values for testing H_0 versus H_1 and H_1 versus H_2 with $\mu_1,\mu_2,\ldots,\mu_k$ normal means. (Some particular partial orders will be discussed in this section.) Robertson and Wegman (1978) show that the chi-bar-squared distributions arise as large sample approximations for the LRTs of trends in parameters in an exponential family. Robertson (1978) considers testing for trends in multinomial parameters. Also, nonparametric tests for trends can be obtained by replacing each observation by its ranking among all the observations (cf. Shirtey (1977)). Chi-bar-squared distributions arise in this setting, too. The bounds established here can be applied in each of these three cases.

For a total order and the simple tree, such bounds were useful in determining approximations for the $\bar{\chi}^2$ and $\bar{E}^2$ distributions, in studying the robustness of the equal-weights ($w_1 = w_2 = \ldots = w_k$) $\bar{\chi}^2$ and $\bar{E}^2$ distributions and for identifying weight sets for which the equal-weights distributions might not provide very accurate approximations. It is anticipated that the bounds given here will be used similarly. For instance, in the case of a simple loop, we see from the proof of Theorem 2.4 that if w_1 and w_k are

exterior elements, which is at least two since Γ is finite with at least two elements. If Γ is indecomposable with no interior elements and $e = 2$, then $P(1,2;w) = P(2,2;w) = \frac{1}{2}$.

Hence, we suppose that the conclusion holds for indecomposable sets with no interior elements and e-1 exterior elements. Consider Γ, an indecomposable partially ordered set with no interior elements and e exterior elements. By Lemma 2.7 there exists $j \in \Gamma$ with $\Gamma - \{j\}$ indecomposable, and it is easy to see that $\Gamma - \{j\}$ has no interior elements and e-1 exterior elements. For convenience relabel so that $j = k$. Because Γ is indecomposable, j is not isolated, and by Lemma 2.3, we can, by making w_k small, make $\{P(\ell,k;w)\}$ uniformly close to the convolution of $\{P(\ell,k-1;w^{(k-1)})\}$ with $\{\frac{1}{2} I_{\{0,1\}}(\ell)\}$. By the inductive hypothesis $\{P(\ell,k-1;w^{(k-1)})\}$ can be made uniformly close to $\{A(\ell,e-1)\}$ and the desired conclusion follows.

Returning to the primary induction, we suppose that the desired conclusion holds for indecomposable sets with i-1 interior elements and that Γ is indecomposable with i interior elements and e exterior elements (of course, $e+i = k$). For convenience suppose that k is an interior element. By Lemma 2.6, we can, by choosing w_k sufficiently small, make $P(\ell,k;w)$ close to $P(\ell,k-1;w^{(k-1)})$. Next, we show that $\Gamma - \{k\}$ is indecomposable. If not there exist A and B with $A \hat{+} B = \Gamma - \{k\}$. There must be elements in both A and B which are comparable with k, for if not, say B contains no elements comparable with k, then $(A \cup \{k\}) \hat{+} B = \Gamma$. This is a contradiction. Suppose $a \in A$ and $b \in B$ with a and b both comparable with k. Now, either $a \ll k$ and $b \ll k$ or $k \ll a$ and $k \ll b$. We consider the case $a \ll k$ and $b \ll$

Proof. The proof is by induction. If $k=2$, the conclusion is obvious. Consider $k>2$. If $\Gamma - \{k\}$ is decomposable, then there exist A and B with $A \hat{+} B = \Gamma - \{k\}$. We will show that $A \cup \{k\}$ is indecomposable (throughout the proof we mean indecomposable wrt the appropriate restriction of <<). If not there exist A' and B' with $A' \hat{+} B' = A \cup \{k\}$. Without loss of generality, we assume $k \notin A'$. Since $A' \subset A$, we know that every element in A' is noncomparable with every element in B. Because every element of A' is noncomparable with every element in B', $A' \hat{+} (B \cup B') = \Gamma$, a contradiction. By the inductive hypothesis, there exists $j \in A \cup \{k\}$ with $A \cup \{k\} - \{j\}$ indecomposable. Next, we show that $\Gamma - \{j\}$ is indecomposable. If not $A'' \hat{+} B'' = \Gamma - \{j\}$. Considering separately the cases $j = k$ and $j \neq k$, we see that $(\Gamma - \{j\}) \cap (A \cup \{k\} - \{j\}) \neq \phi$. For convenience, let $A'' \cap (A \cup \{k\} - \{j\}) \neq \phi$. Because $A \cup \{k\} - \{j\}$ is indecomposable, $A'' \supset A \cup \{k\} - \{j\}$. It is not difficult to show that $B'' \subset B$. Every element in B, and consequently every element in B'', is noncomparable with j. Also, every element in A'' is noncomparable with every element in B''. Hence, $(A'' \cup \{j\}) \hat{+} B'' = \Gamma$, which is a contradiction. ■

Theorem 2.8. If $(\Gamma, <<)$ is indecomposable with e exterior elements, then there exists a sequence of positive weights $w(n) = (w_1(n), w_2(n), \ldots, w_k(n))$ for which

$$\lim_{n \to \infty} P(\ell, k; w(n)) = A(\ell, e) \text{ for } \ell = 1, 2, \ldots, k.$$

Of course, $A(\ell, e) = 0$ for $\ell > e$.

Proof. The proof is an induction on the number of interior elements. We first show that the conclusion is valid for indecomposable Γ with no interior elements. This part of the proof is an induction on the number of

C_{i_0}. This produces a partition in $\mathcal{L}^{(2)}_{\ell k}$, and the probability that this partition occurs converges, as $w_k \to 0$, to the probability that $\{C_1, C_2, \ldots, C_\ell\}$ occurs. Furthermore, including k in any other C_i or adding the singleton $\{k\}$ will not produce a partition in $\mathcal{L}_{\ell k}$ or $\mathcal{L}_{\ell+1,k}$, respectively. If $i_0 < i_1$ then $\{C_1, C_2, \ldots, C_\ell\}$ can be made a partition of Γ by including the singleton $\{k\}$ between C_{i_0} and C_{i_1} or by adding k to one of $C_{i_0}, C_{i_0+1}, \ldots, C_{i_1}$. But, we have seen that each of these partitions of Γ have limit 0 as $w_k \to 0$, except for the two that include k either with C_{i_0} or C_{i_1}. The sum of the probabilities of these two decompositions approaches $P(\{C_1, C_2, \ldots, C_\ell\})$ as $w_k \to 0$. Hence, $P(\ell,k;w) \to P(\ell,k-1;w^{(k-1)})$ as $w_k \to 0$, and the proof is completed.

We now return to the consideration of the sharpness of the results in Theorem 2.5. More will be established; the assumption that Γ has at most one minimal or at most one maximal element will be relaxed. However, as we have seen, the bounds given there are not sharp if Γ is decomposable. We will show that if Γ is indecomposable, then the bounds given in Theorem 2.5 can be obtained as the limit of $P(\ell,k;w_n)$ as $n \to \infty$. This shows that the bounds given in Theorem 2.5, as well as those derived from them by convolutions, are sharp. Another lemma is needed.

<u>Lemma 2.7</u>. If $(\Gamma, \ll)$ is indecomposable with card. $(\Gamma) > 1$, then there exists a $j \in \Gamma$ with $\Gamma - \{j\}$ indecomposable wrt the restriction of $\ll$ to $\Gamma - \{j\}$.

element in B_j, then $\phi \nsubseteq (L-L_{j-1}) \cap B_j \nsubseteq B_j$ implies that $\phi \nsubseteq (L-L_{j-1}) \cap (B_j-\{k\}) \nsubseteq B_j-\{k\}$ and hence,

$$P(1,\text{card.}\ (B_j);w(B_j)) \to P(1,\text{card.}\ (B_j-\{k\});w(B_j-\{k\}))\ \text{as}\ w_k \to 0.$$

Thus, $P(\{B_1,B_2,\ldots,B_\ell\}) \to P(\{B_1,B_2,\ldots,B_j-\{k\},\ldots,B_\ell\})$ as $w_k \to 0$.

Finally, we consider decompositions in $\mathcal{L}^{(3)}_{\ell k}$. If k is a minimal, but not an isolated element by B_j, then $\phi \nsubseteq (L-L_{j-1}) \cap B_j \nsubseteq B_j$ implies that $(L-L_{j-1}) \cap (B_j-\{k\}) \nsubseteq B_j-\{k\}$, but recall $L_{j-1} \cup \{k\} \varepsilon \mathcal{L}$. For $Z_k < 0$, $Z_k/\sqrt{w_k} \to -\infty$ and so B_j would not be a level set for w_k sufficiently small. Hence, as $w_k \to 0$,

$$P(1,\text{card.}\ (B_j);w(B_j)) \to P(Z_k>0)P(1,\text{card.}(B_j-\{k\});w(B_j-\{k\})).$$

If k is a maximal, but not an isolated element of B_j, then a similar argument shows that as $w_k \to 0$,

$$P(1,\text{card.}\ (B_j);w(B_j) \to P(Z_k<0)P(1,\text{card.}(B_j-\{k\});w(B_j\{k\})).$$

Because $\mathcal{L}_{\{1,2,\ldots,k-1\}} = \{L \cap \{1,2,\ldots,k-1\}: L\ \varepsilon \mathcal{L}\}$, every partition of Γ, $\{B_1,B_2,\ldots,B_\ell\}$, can be made a partition of $\{1,2,\ldots,k-1\}$ by deleting k from the B_j containing it. Conversely, let $\{C_1,C_2,\ldots,C_\ell\}\ \varepsilon\ \mathcal{L}_{\ell,k-1}$. Set $i_0 = \max\{i$: there is an $\alpha\ \varepsilon\ C_i$ with $\alpha << k\}$ and $i_1 = \min\{i$: there exists $\beta\ \varepsilon\ C_i$ with $k << \beta\}$. (Recall, k is an interior element.) Now, $i_0 \le i_1$. For, if $i_1 < i_0$, then $C_1 \cup \ldots \cup C_{i_1} \notin \mathcal{L}_{\{1,2,\ldots,k-1\}}$, which is a contradiction. If $i_0 = i_1$, then $\{C_1,C_2,\ldots,C_\ell\}$ can be made a partition of Γ by including k in

decompositions of Γ in which k is an exterior element of B_j but not an isolated element in B_j. We now show that the sum in (2.9) over $\mathcal{L}^{(1)}_{\ell k}$ converges to zero as $w_k \to 0$. There are two cases to consider, $B_j = \{k\}$ and card.$(B_j) \geq 2$. If $B_j = \{k\}$, then because k is an interior element in Γ, $1 < j < \ell$. As $w_k \to 0$, $\mathrm{Var}(\mathrm{Av}(B_1))$ and $\mathrm{Var}(\mathrm{Av}(B_\ell))$ are fixed, but $\mathrm{Var}(\mathrm{Av}(B_j)) \to \infty$. Hence, $P(\mathrm{Av}(B_1) < \ldots < \mathrm{Av}(B_\ell)) \leq P(\mathrm{Av}(B_1) < \mathrm{Av}(B_j) < \mathrm{Av}(B_\ell)) \to 0$. (See Robertson and Wright (1982, p. 305).) This also shows that $P(k,k;w) \to 0$ as $w_k \to 0$. We next consider the case in which k is an isolated element of B_j and card. $(B_j) \geq 2$, but first we note that the event in which there is one level set in the projection of $\bar{X}(B_j)$ onto the cone determined by $\ll_{B_j}$ is determined by

$$\mathrm{Av}((L-L_{j-1}) \cap B_j) \geq \mathrm{Av}(B_j) \text{ for all } L \in \mathcal{L} \text{ with } \phi \neq (L-L_{j-1}) \cap B_j \neq B_j,$$

and that the probability of equality is zero. If k is isolated in B_j and card. $(B_j) \geq 2$, then $L_{j-1} \cup \{k\}$, $L_j - \{k\} \in \mathcal{L}$ (if $\alpha \ll k$, then $\alpha \in L_{j-1}$ or $\alpha \in B_j$, but k is isolated in B_j; and if $k \ll \alpha$, then $\alpha \in L^c_{j-1}$). Thus, $P(1,$ card. $(B_j); w(B_j) \leq$

$$P(\bar{X}_k \geq \mathrm{Av}(B_j) \text{ and } \mathrm{Av}(B_j-\{k\}) \geq \mathrm{Av}(B_j)) = P(\bar{X}_k \geq \mathrm{Av}(B_j-\{k\}) \geq \bar{X}_k) = 0.$$

Next, we consider partitions in $\mathcal{L}^{(2)}_{\ell k}$. Because, $B_j - \{k\} \neq \phi$, $P(\mathrm{Av}(B_1) < \mathrm{Av}(B_2) < \ldots < \mathrm{Av}(B_\ell)) \to P(\mathrm{Av}(B_1) < \ldots < \mathrm{Av}(B_j-\{k\}) < \ldots < \mathrm{Av}(B_\ell))$ as $w_k \to 0$. Replacing $\bar{X}_k$ by $Z_k/\sqrt{w_k}$, it is easily seen that if k is an interior

with $P(\ell,k-1;w^{(k-1)})$ the probability, under $\mu_1 = \mu_2 = \ldots = \mu_{k-1}$, that $\bar{\mu}^{(k-1)}$ has exactly ℓ distinct values. (Of course, $P(k,k-1;w^{(k-1)}) = 0$.)

Proof. Assume $\mu \in H_0$. As we have seen, the MLSA decomposes Γ into disjoint sets $B_1,B_2,\ldots,B_h$, with the B_i successive differences of a nondecreasing collection of lower layers. For $1 \leq \ell \leq k$, let $\mathcal{L}_{\ell k}$ be the collection of all such decompositions with $h = \ell$. For $\{B_1,B_2,\ldots,B_\ell\} \in \mathcal{L}_{\ell k}$, let $w(B_i)$ $(\bar{X}(B_i))$ be the vector of weights w_j (means $\bar{X}_j$) with $j \in B_i$, $1 \leq i \leq \ell$. Barlow et al. (1972, eq (3.23)) proved that $P(\ell,k;w) =$

(2.9 $\sum_{\{B_1,B_2,\ldots,B_\ell\}\in\mathcal{L}_{\ell k}} P(\{B_1,B_2,\ldots,B_\ell\})$

with $P(\{B_1,B_2,\ldots,B_\ell\}) =$

$$P(Av(B_1) < Av(B_2) < \ldots < Av(B_\ell)) \prod_{i=1}^{\ell} P(1,\text{card.}(B_i);w(B_i)).$$

Note that for any such decomposition, if $\alpha \ll \beta$ and $\beta \in B_i$, then $\alpha \in B_1 \cup B_2 \cup \ldots \cup B_i$, and if $\alpha \ll \beta$ and $\alpha \in B_i$, then $\beta \in B_i \cup \ldots \cup B_\ell$. Further note that such a decomposition does not occur as the level sets with positive probability if $\prod_{i=1}^{\ell} P(1,\text{card.}(B_i);w(B_i)) = 0$.

Let j denote the index for which $k \in B_j$. Partition $\mathcal{L}_{\ell k}$ into the following three parts: $\mathcal{L}_{\ell k}^{(1)}$ consists of those decompositions of Γ in which k is an isolated element of B_j, $\mathcal{L}_{\ell k}^{(2)}$ consists of those decompositions of Γ in which k is an interior element of B_j and $\mathcal{L}_{\ell k}^{(3)}$ consists of those

isotonic wrt $<<'\} \subset H_1$, and replacing H_1 by H_1' provides a lower (upper) bound for $\bar{\chi}^2_{01}$ and $\bar{E}^2_{01}(\bar{\chi}^2_{12}$ and $\bar{E}^2_{12})$. As we have seen earlier in this section, bounds for the quasi order, $<<'$, can be obtained from bounds for its reduction $<<'^*$. However, $<<'^*$ is a simple tree on $\{1,2,\ldots,e\}$. By Theorem 1 of Wright and Tran (1985), the appropriate bounds for this simple tree are obtained with mixing coefficients $\binom{e-1}{\ell-1}2^{-e+1}$, which is our $A(\ell,e)$. The proof is completed by applying the comment given after the proof of the Remark in this section.

If $\Gamma = \Gamma_1 \hat{+} \Gamma_2 \hat{+} \cdots \hat{+} \Gamma_d$ and each Γ_α satisfies the hypotheses of Theorem 2.5 with e_α the number of exterior elements, then the convolution of the $\{A(\ell,e_\alpha)\}$ for $\alpha = 1,2,\ldots,d$ can be used to provide bounds analogous to (2.8a,b,c,d). However, this convolution is $A_d(\ell,e)$ with $e = e_1 + e_2 + \cdots + e_d$. Now, e is the number of exterior elements in Γ, but $\{A(\ell,e)\}$ and $\{A_d(\ell,e)\}$ are not the same unless $d = 1$. In fact, if $d > 1$ and $c > 0$ the right hand side of (2.8 a,b) ((2.8 c,d)) is larger (smaller) if $A(\ell,e)$ is replaced by $A_d(\ell,e)$. This shows that if one first decomposed Γ, then better lower (upper) bounds for $\bar{\chi}^2_{01}$ and $\bar{E}^2_{01}$ ($\bar{\chi}^2_{12}$ and $\bar{E}^2_{12}$) are obtained.

The proof of the sharpness of these bounds rests on the following.

Lemma 2.6. If k is an interior element in $(\Gamma,<<)$, then

$$\lim_{w_k \to 0} P(\ell,k;w) = P(\ell,k-1;w^{(k-1)})$$

coefficients which provide upper (lower) bounds for the Γ_α, then their convolutions provide upper (lower) bounds for Γ. However, if the breadth of Γ_α is b_α then it is easy to show that $b = b_1 + b_2 + \dots + b_d$ is the breadth of Γ and the convolution of the $\{A_{b_\alpha}(\ell, \mathrm{card}(\Gamma_\alpha)-b_\alpha)\}$ is $\{A_b(\ell,k)\}$. Thus it is not necessary to decompose Γ for the results in Theorem 2.2. But, as we shall see, it is necessary to decompose Γ to obtain sharp bounds of the type provided in the next theorem. It should be noted that the hypotheses of Theorem 2.5 imply that Γ is indecomposable.

Theorem 2.5. Let $(\Gamma,\ll)$ be a partially ordered set which has at most one maximal element or at most one minimal element and let e denote its number of exterior elements. For $\mu \in H_0$ and any w with $w_i > 0$ for all $i \in \Gamma$,

(2.8a) $P_w[\bar{\chi}^2_{01} \geq c] \geq \sum_{\ell=1}^{e} A(\ell,e)P[\chi^2_{\ell-1} \geq c]$,

(2.8b) $P_w[\bar{E}^2_{01} \geq c] \geq \sum_{\ell=1}^{e} A(\ell,e)P[B_{\frac{1}{2}(\ell-1),\frac{1}{2}(N-\ell)} \geq c]$,

(2.8c) $P_w[\bar{\chi}^2_{12} \geq c] \leq \sum_{\ell=1}^{e} A(\ell,e)P[\chi^2_{k-\ell} \geq c]$, and

(2.8d) $P_w[\bar{E}^2_{12} \geq c] \leq \sum_{\ell=1}^{e} A(\ell,e)P[B_{\frac{1}{2}(k-\ell),\frac{1}{2}(N-k)} \geq c]$.

Proof. We give the proof for the case in which Γ has at most one minimal element. The proof for the other case is similar. Because Γ is finite, it has exactly one minimal element. We suppose that the minimal element is 1 and the maximal elements are $2,3,\dots,e$. Let $\ll'$ be the quasi order which requires $1 = e+1 = e+2 = \dots = k \ll' j$ for $j = 2,3,\dots,e$. If $x \in R^k$ is isotonic wrt $\ll'$, then it is also isotonic wrt $\ll$. Hence, $H_1' = \{x \in R^k\colon x$

is an exterior element which is not isolated. Now $\Gamma - \{k\}$ with the restriction of $<<$ has breadth b, and by the inductive hypothesis there is a weight set $w'(\varepsilon))$ for which $|P(\ell,k-1;w'(\varepsilon)) - A_b(\ell,k-1)| < \varepsilon/2$ for $\ell = 1,2,\ldots,k-1$. Set $w(\varepsilon) = (w'(\varepsilon),w_k)$. By Lemma 2.3, for w_k small enough, $|P(\ell,k;w(\varepsilon)) - \frac{1}{2}P(\ell,k-1;w'(\varepsilon)) - \frac{1}{2}P(\ell-1,k-1;w'(\varepsilon))| < \varepsilon/2$. Using the fact that $A_b(\ell,k) = \frac{1}{2}A_b(\ell-1,k-1) + \frac{1}{2}A_b(\ell,k-1)$, we see that $|P(\ell,k;w(\varepsilon)) - A_b(\ell,k)|$

$$\leq |P(\ell,k;w(\varepsilon))-\frac{1}{2}P(\ell,k-1;w'(\varepsilon))-\frac{1}{2}P(\ell-1,k-1;w'(\varepsilon))|$$

$$+\frac{1}{2}|P(\ell,k-1;w'(\varepsilon))-A_b(\ell,k-1)| + \frac{1}{2}|P(\ell-1,k-1;w'(\varepsilon))-A_b(\ell-1,k-1)| < \varepsilon.$$

The proof is completed.

The next result provides lower (upper) stochastic bounds for the null distribution of $\bar{\chi}^2_{01}$ and $\bar{E}^2_{01}$ ($\bar{\chi}^2_{12}$ and $\bar{E}^2_{12}$) for a large class of partial orders of practical importance. This class includes all the partial orders discussed in the Introduction.

If a partially ordered set, $(\Gamma,<<)$, can be partitioned into two nonempty, disjoint subsets A and B with i and j noncomparable for every $i \in A$ and every $j \in B$, then we write $A \mathbin{\hat{+}} B = \Gamma$ and say that Γ is <u>decomposable</u>. Otherwise, it is <u>indecomposable</u>. If Γ can be written as the union of nonempty disjoint subsets $\Gamma_1,\Gamma_2,\ldots,\Gamma_d$ with the elements of Γ_α and Γ_β noncomparable for all $1 \leq \alpha \neq \beta \leq d$ (ie. $\Gamma_1 \mathbin{\hat{+}} \Gamma_2 \mathbin{\hat{+}} \cdots \mathbin{\hat{+}} \Gamma_d = \Gamma$), then the mixing coefficients for Γ are the convolution of those for $\Gamma_1,\Gamma_2,\ldots,$ and Γ_d (cf. Barlow et al. (1972, p. 148)). Hence, if one obtains the mixing

comparable with k. (Such exists by hypothesis.) For w_k sufficiently small, $Z_k/\sqrt{w_k} > \max(\bar{X}_1,\ldots,\bar{X}_{k-1})$. Using the MLSA to compute the projection of $(\bar{X}^{(k-1)}, Z_k/\sqrt{w_k})$, we see that the first i-1 level sets are $B_1,B_2,\ldots,B_{i-1}$. Neglecting the set with probability zero on which Av(A) = Av(B) for some A and B distinct subsets of Γ and noting that $Av(A \cup \{k\}) \to Av(A)$ as $w_k \to 0$ for $\phi \neq A \subset \Gamma - \{k\}$, we see that for sufficiently small w_k, $B_i \cup \{k\}$ is the next level set. Hence, for $Z_k > 0$ and such w_k, $M(\bar{X}^{(k-1)}, Z_k/\sqrt{w_k};w) = M(\bar{X}^{(k-1)};w^{(k-1)})$. The proof is completed.

Theorem 2.4. Let $(\Gamma,<<)$ be a partially ordered set with breadth b. There exists a sequence of positive weight sets $w(n) = (w_1(n),\ldots,w_k(n))$ for which $\lim_{n\to\infty} P(\ell,k;w(n)) = A_b(\ell,k)$ for $\ell = 1,2,\ldots k$.

Proof. The proof is an induction on k-b. If k-b = 0, then the partial order is trivial, there are no comparable elements. Hence, $\bar{\mu} = \bar{X}$.

Suppose that k-b > 0 and by relabelling if necessary, let $\{1,2,\ldots,b\}$ be a maximal antichain. We show that for an arbitrary $\epsilon > 0$, there exists a weight set $w(\epsilon)$ with $|P(\ell,k;w(\epsilon)) - A_b(\ell,k)| < \epsilon$ for $\ell = 1,2,\ldots,k$. None of the elements $b+1,b+2,\ldots,k$ are isolated because $\{1,2,\ldots,b\}$ is a maximal antichain. Furthermore, we will show that at least one of $b+1,b+2,\ldots,k$ is an exterior element. If not, then they are all interior elements and one can construct a chain containing b+1 with both its smallest and largest elements in $\{1,2,\ldots,b\}$. But, this is a contradiction since no two elements in $\{1,2,\ldots,b\}$ are comparable. Without loss of generality we assume that k

projection of $\bar{X}$ ($\bar{X}^{(k-1)}$) onto the cone determined by << (<< restricted to Γ - {j}).

<u>Lemma 2.3</u>. Let $\mu_1 = \mu_2 = \ldots \mu_k = \mu$. If k is an exterior element but not isolated, then

$$M(\bar{X};w) \overset{\mathcal{D}}{\to} M(\bar{X}^{(k-1)};w^{(k-1)}) + I_{(-\infty,\mu)}(\bar{X}_k)$$

as $w_k \to 0$.

<u>Proof</u>. Without loss of generality, assume $\mu = 0$. We suppose k is a minimal element. The proof for a maximal element is similar. By enlarging the probability space if necessary, we can obtain $Z_k \sim N(0,\sigma_0^2)$ independent of $\bar{X}_1,\ldots,\bar{X}_{k-1}$, and of course, $M(\bar{X};w) \overset{\mathcal{D}}{=} M(\bar{X}^{(k-1)},Z_k/\sqrt{w_k};w)$. For a fixed element in the underlying probability space, we consider the cases $Z_k < 0$ and $Z_k > 0$. If $Z_k < 0$, then for sufficiently small w_k, $Z_k/\sqrt{w_k} < \min(\bar{X}_1,\ldots,\bar{X}_{k-1})$. Using the MLSA and the fact that {k} is a lower layer, we see that the first level set of the projection of $(\bar{X}^{(k-1)},Z_k/\sqrt{w_k})$ is {k} and the others are the level sets of $\bar{\mu}^{(k-1)}$, the projection of $\bar{X}^{(k-1)}$ with weights $w^{(k-1)}$. Hence, for sufficiently small w_k,

$$M(\bar{X}^{(k-1)}, Z_k/\sqrt{w_k};w) = M(\bar{X}^{(k-1)};w^{(k-1)}) + 1,$$

Next, we consider the case $Z_k > 0$. Let $B_1,\ldots,B_h$ denote the level sets for $\bar{\mu}^{(k-1)}$ and let B_i be the first level set containing an element

$Y_b + b$ is a stochastic upper bound for M and $P[Y_1 + \ldots + Y_b + b = \ell] = \binom{k-b}{\ell-b}2^{-k+b} = A_b(\ell,k)$. The proof is completed.

We next show that the bounds in Theorem 2.2 are sharp. Before giving the proof, the following definitions are needed.

Definitions. An element $i \in \Gamma$ is said to be maximal (minimal) if there does not exist $j \in \Gamma$ with $i \ll j$ ($j \ll i$). Furthermore, i is said to an exterior element if it is a maximal or minimal element and otherwise, it is an interior element. An element which is both maximal and minimal is called isolated.

The minimum lower sets algorithm (MLSA) for computing projections onto a cone of isotonic functions is used in the proof of the lemma below. A subset, $L \subset \Gamma$, is a lower layer provided $i \in \Gamma$ implies that $j \in \Gamma$ for all $j \ll i$. Let $\mathcal{L}$ denote the collection of lower layers in Γ. For $A \subset \Gamma$, define $Av(A) = \sum_{i\in A} w_i\bar{X}_i/\sum_{i\in A} w_i$ and set $L_0 = \phi$. Given $L_0 \subset L_1 \subset \ldots \subset L_j$, the MLSA chooses L_{j+1} to properly contain L_j and minimize $Av(L-L_j)$ over all such lower layers L, and in fact, L_{j+1} is chosen to be the largest such minimizer. This procedure terminates with $L_h = \Gamma$. If $i \in L_{j+1} - L_j \equiv B_j$, then $\bar{\mu}_i = Av(B_j)$. The B_j are the level sets for $\bar{\mu}$ (cf. Barlow et al. (1972, p. 76)).

Let $\bar{X}^{(k-1)} = (\bar{X}_1,\bar{X}_2,\ldots,\bar{X}_{k-1})$, $w^{(k-1)} = (w_1,w_2,\ldots,w_{k-1})$ and $M(\bar{X};w)$ $(M(\bar{X}^{(k-1)};\ w^{(k-1)}))$ be the number of distinct values in the

$H_1' = \{x \in R^k$: x is isotonic wrt $\ll'\}$. Examining the definitions of $\bar{X}^2_{01}, \bar{X}^2_{12}, \bar{E}^2_{01}$ and $\bar{E}^2_{12}$ and noting that $\bar{E}^2_{12}$ is an increasing function of $\|\bar{X} - \bar{\mu}\|^2_w/Q$, we see that an upper (lower) bound for $\bar{X}^2_{01}$ and $\bar{E}^2_{01}$ ($\bar{X}^2_{12}$ and $\bar{E}^2_{12}$) is obtained by replacing H_1 by H_1'. Thus, we need to find the appropriate bounds for the cone H_1'. Robertson and Wright (1982) have shown that because the chi-squared and beta variables involved in (2.1) and (2.2) ((2.3) and (2.4)) are stochastically increasing (decreasing) with ℓ, we only need to find $D(\ell,k)$ with $\sum_{\ell=j}^k P(\ell,k;w) \le \sum_{\ell=j}^k D(\ell,k)$ for $j = 1,2,\ldots,k$. If M is the random variable which denotes the number of distinct elements in $E_w(\bar{X}|H_1')$, we seek a stochastic upper bound for M. Since $\sum_{i=1}^k w_i(\bar{X}_i - \mu_i)^2 = \sum_{\alpha=1}^b \sum_{i\in\Gamma_\alpha} w_i(\bar{X}_i - \mu_i)^2$, we may obtain the projection of $\bar{X}$ onto H_1' by projecting the subvector of $\bar{X}$ with coordinates indexed by elements of Γ_α onto the cone which requires such vectors to be isotonic wrt the restriction of $\ll'$ to Γ_α for $\alpha = 1,2,\ldots,b$. Let M_α denote the number of distinct elements in the αth such projection for $\alpha = 1,2,\ldots,b$. Clearly, $M_1, M_2, \ldots, M_b$ are independent and, with probability one, $M = M_1 + M_2 + \ldots + M_b$. Assume $\mu \in H_0$. Let $Y_\alpha \sim B(\text{card}.(\Gamma_\alpha) - 1, \frac{1}{2})$ ($B(0,\frac{1}{2})$ is degenerate at 0) for $1 \le \alpha \le b$ with $Y_1, Y_2, \ldots, Y_b$ independent. Because $\ll'$ restricted to Γ_α is a total order, $Y_\alpha + 1$ is a stochastic upper bound for M_α for $\alpha = 1,2,\ldots,b$. Hence, $Y_1 + \ldots +$

"small" and $w_2, \ldots, w_{k-1}$ are not, then the $P(\ell,k;w)$ will be close to $A_{k-2}(\ell,k)$ and the corresponding bounds would provide better approximations than the equal-weights ones. On the other hand, if $w_2, \ldots, w_{k-1}$ are small and w_1 and w_k are not, then $P(\ell,k;w)$ will be close to $A(\ell,2)$. (See the proof of Theorem 2.8.)

If Γ has at most one maximal or at most one minimal element, then, applying Theorems 2.2 and 2.5, we see that for $c > 0$ and $\mu \in H_0$,

$$\sum_{\ell=1}^{e}\binom{e-1}{\ell-1}2^{1-e}P[\chi^2_{\ell-1} \geq c] \leq P_w[\bar{\chi}^2_{01} \geq c] \leq \sum_{\ell=b}^{k}\binom{k-b}{\ell-b}2^{b-k}P[\chi^2_{\ell-1} \geq c]$$

and

$$\sum_{\ell=b}^{k}\binom{k-b}{\ell-b}2^{b-k}P[\chi^2_{k-\ell} \geq c] \leq P_w[\bar{\chi}^2_{12} \geq c] \geq \sum_{\ell=1}^{e}\binom{e-1}{\ell-1}2^{1-e}P[\chi^2_{k-\ell} \geq c],$$

with e the number of exterior elements in Γ and b the breadth of Γ. For a simple tree, $e = k$ and $b = k-1$; for a simple loop, $e = 2$ and $b = k-2$; for a unimodal ordering (ie. $1 \ll 2 \ll \ldots \ll h \gg h+1 \gg \ldots \gg k$), $e = 3$ and $b = 2$; and for the ordering on $\Gamma = \{(i,j): 1 \leq i \leq R,\ 1 \leq j \leq c\}$ given by $(i,j) \ll (s,t)$ provided $i \leq s$ and $j \leq t$, $e = 2$ and $b = \min(R,C)$.

We indicate one final application. Suppose that $U_1, U_2, \ldots, U_m$ are ordered categorical variables of interest on a certain population and that large values of each are desirable. Further, suppose that one wishes to determine whether or not a new treatment tends to produce values of $U_1, U_2, \ldots, U_m$ which are at least as large as under a control, that is whether or not the treatment distribution is stochastically larger than the control. LRTs for the case $m = 1$ are discussed in Robertson and Wright (1981). Let $p_{i_1 i_2 \ldots i_m}$ ($q_{i_1 i_2 \ldots i_m}$) denote the proportion of the population with U_α at

level i_α, $1 \le \alpha \le m$, under the treatment (control); let $<<$ be the coordinatewise ordering on $\Gamma = \{(i_1, i_2, \ldots, i_m): 1 \le i_\alpha \le I_\alpha,\ 1 \le \alpha \le m\}$; and let denote the lower layers wrt $<<$. Robertson and Wright (1974) studied the multivariate concept of stochastic ordering which requires

$$H: \sum_{(i_1,i_2,\ldots,i_m) \varepsilon L} p_{i_1 i_2 \ldots i_m} \le \sum_{(i_1,i_2,\ldots,i_m) \varepsilon L} q_{i_1 i_2 \ldots i_m}$$

for each $L \varepsilon \mathcal{L}$. H is equivalent to requiring $f(U_1, U_2, \ldots, U_m)$ to be stochastically larger under p than under q for all $f: R^m \to R$ which are nondecreasing in each variable with the others fixed. Let Λ denote the LR for testing H versus $\sim H$ based on independent samples and $T = -2 \ln \Lambda$. It can be shown that the significance level corresponding to t, an observed value of T, ie. $\sup_{p,q\varepsilon H} P_{p,q}[T \ge t]$, is for large samples approximately

$$(3.1) \quad \sup_p \sum_{\ell=1}^{k} P(\ell,k;p) P[\chi^2_{\ell-1} \ge t] \text{ with } k = I_1 I_2 \ldots I_m.$$

(The details will be given elsewhere.) The breadth of this Γ is $b = \min(I_1, I_2, \ldots, I_m)$, and applying Theorem 2.2, we see that (3.1) is

$$\sum_{\ell=b}^{k} \binom{k-b}{\ell-b} 2^{b-k} P[\chi^2_{\ell-1} \ge t].$$

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REFERENCES

Barlow, R.E.,Bartholomew, D.J.,Bremner, J.M. and Brunk, H.D. (1972). Statistical Inference Under Order Restrictions.Wiley.New York.

Bartholomew, D.J. (1959)."A Test of Homogeneity for Ordered Alternatives I, II."Biometrika 46,36-48,328-335.

Bartholomew, D.J. (1961)."A Test for Homogeneity of Means Under Restricted Alternatives (with discussion)."J. Roy. Statist. Soc.B 18,234-239.

Brunk, H.D.,Ewing, G.M. and Utz, W.R. (1957). "Minimizing Integrals in Certain Classes of Monotone Functions."Pacif. J. Math. 7,833-847.

Chase, G.R. (1974)."On Testing for Ordered Alternatives with Increased Sample Size for a Control."Biometrika 61,569-578

Crawley, P. and Dilworth, R.P. (1973).Algebraic Theory of Lattices,Prentice-Hall,Englewood Cliffs,New Jersey.

Dykstra, R.L. (1981)."An Isotonic Regression Algorithm."J. Statist. Planning Inference 5,355-363.

Dykstra, R.L. and Robertson, Tim (1982)."Order Restricted Statistical Tests on Multinomial and Poisson Parameters:The Starshaped Restriction." Ann. Statist. 10,1246-1252.

Dykstra, R.L. and Robertson, Tim (1983)."On Testing Monotone Tendencies." J. Amer. Statist. Assn. 78,342-350.

Hanson, D.L.,Pledger,Gordon and Wright, F.T. (1973)."On Consistency in Monotonic Regression."Ann. Statist. 1,401-421.

Kudô , A. (1963)."A Multivariate Analogue of the One-sided Test." Biometrika 50,403-418.

Nüesch, P.E. (1966)."On the Problem of Testing Location in Multivariate Populations for Restricted Alternatives."Ann. Math. Statist. 37, 113-119.

Perlman, M.D. (1969)."One-sided Problems in Multivariate Analysis."Ann. Math. Statist. 40,549-567.

Robertson, Tim (1978)."Testing for and Against an Order Restriction in Multinomial Parameters."J. Amer. Statist. Assn. 73,197-202.

Robertson, Tim and Wegman, E.J. (1978)."Likelihood Ratio Tests for Order Restrictions in Exponential Families."Ann. Statist. 6,485-505.

Robertson, Tim and Wright, F.T. (1974)."On the Maximum Likelihood Estimation of Stochastically Ordered Random Variates."Ann. Statist. 2, 528-534.

Robertson, Tim and Wright, F.T. (1981)."Likelihood Ratio Tests For and Against a Stochastic Ordering Between Multinomial Populations."Ann. Statist. 9,1248-1257.

Robertson, Tim and Wright, F.T. (1982)."Bounds on Mixtures of Distributions Arising in Order Restricted Inference."Ann. Statist. 10,302-306.

Robertson, Tim and Wright, F.T. (1983)."On Approximation of the level Probabilities and Associated Distributions in Order Restricted Inference."Biometrika 70,597-606.

Robertson, Tim and Wright, F.T. (1985)."One-sided Comparisons for a Treatment with a Control."Can. J. Statist. 13,109-122.

Shirley, E. (1977)."A Nonparametric equivalent of William's Test for Contrasting Increasing Dose Levels of a Treatment."Biometrics 33, 386-389.

Shorack, G.R. (1967)."Testing Against Ordered Alternatives in Model I Analysis of Variance:Normal Theory and Nonparametric."Ann. Math. Statist. 38,1740-1753.

Siskind, V. (1976)."Approximate Probability Integrals and Critical Values for Bartholomew's Test for Ordered Means."Biometrika 63,647-654.

Wright, F.T. and Tran, T.A. (1985)."Approximating the Level Probabilities in Order Resticted Inference:The Simple Tree Ordering."Biometrika 72, 429-440.

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